

The Lee Fields Medal VII: SOLUTIONS

1. Show that there are no strictly positive integers k , m , and n such that

$$3^{2m} - 2^{2n} = 7^k.$$

Solution: Suppose that $k, m, n \in \mathbb{N}$ is a solution. Note that $3^{2m} - 2^{2n} = (3^m)^2 - (2^n)^2$ is a difference of two squares, so using $x^2 - y^2 = (x - y)(x + y)$:

$$3^{2m} - 2^{2n} = (3^m - 2^n)(3^m + 2^n).$$

By prime decomposition, to be equal to 7^k both must be powers of 7, say $3^m - 2^n = 7^a$ and $3^m + 2^n = 7^b$. Because $3^m + 2^n > 3^m - 2^n$, $b > a$, say $b = a + c$, for $c > 0$. Add these together:

$$\begin{aligned} 2 \cdot 3^m &= 7^a + 7^{a+c} \\ &= 7^a(1 + 7^c) \end{aligned}$$

But the left-hand side does not have any factor 7 so we can conclude $7^a = 1$ and so $3^m - 2^n = 1$. It follows that $2^n = 3^m - 1$ and $3^m + 2^n = 7^k$ which implies that

$$3^m + 3^m - 1 = 7^k \implies 2 \cdot 3^m - 1 = 7^k.$$

Note that the left-hand is one less a multiple of three, so gives remainder two on division by three¹. However the right hand side gives remainder one on division by three because it gives remainder one on division by three. Why? Well,

$$7^1 = 3 \cdot 2 + 1,$$

and if $7^d = 3 \cdot c_d + 1$, then:

$$7^{d+1} = 7 \cdot 3 \cdot c_d + 7 = \underbrace{3 \cdot 7 \cdot c_d + 3 \cdot 2}_{\text{a multiple of three}} + 1.$$

There is no solution.

Statistically this was the joint hardest question with Q.5 — however while two students got full marks on Q. 5, no student got this question correct.

¹ $2 \cdot 3^m - 3 + 3 - 1 = 3(2 \cdot 3^{m-1} - 1) + 2$

Remark: Recall we were led to conclude that if there was a solution $k, m, n \in \mathbb{N}$ that $3^m - 2^n = 1$. A solution to $3^m - 2^n = 1$ is given by $3^2 - 2^3 = 1$. In 1844, a French–Belgian mathematician Catalan guessed that this was the only solution to $3^m - 2^n = 1$ — indeed the only solution to $a^m - b^n = 1$. This was finally proved 158 years later by a Romanian mathematician Mihăilescu. We can then note that at $m = 2$ and $n = 3$, $3^{2 \cdot 2} - 2^{2 \cdot 3} = 3^4 - 2^6 = 17$, which is not a power of 7.

2. Let $a > 0$ be a real number. Calculate

$$\cdots \sqrt{\cdots \sqrt{\sqrt{\sqrt{a}}}},$$

where the number of square roots grows to infinity. Justify your answer.

Solution: Essentially what happens here is that

$$\sqrt{a} \text{ is } \begin{cases} < a, & \text{if } a > 1 \\ = a, & \text{if } a = 1 \\ > a, & \text{if } a < 1. \end{cases}$$

So applying the square root function repeatedly, depending on a , keeps getting you closer and closer to one. To fully justify this you could write:

$$\sqrt{a} = a^{1/2}, \sqrt[2]{\sqrt{a}} = (a^{1/2})^{1/2} = a^{1/4}, \dots, \sqrt[n]{a} = a^{1/2^n},$$

and as $n \rightarrow \infty$, $1/2^n \rightarrow 0$, and so $\sqrt[n]{a} \rightarrow a^0$ and $a^0 = 1$. This requires *continuity* of the function $x \mapsto a^x$.

One student said assume $y = \lim_{n \rightarrow \infty} \sqrt[n]{a}$ and asserted:

$$y = \sqrt{y} \implies y^2 = y \implies y(y - 1) = 0 \implies y = 0 \text{ or } y = 1.$$

I am not sure at this point how to throw out the $y = 0$ possibility without arguing as per the above. Also this approach assumes that the sequence $x_n := \sqrt[n]{a}$ does converge to something — one student appealed to a technical result that an increasing/decreasing sequence that is bounded above/below is convergent.

3. Which real numbers are solutions of the equation:

$$(x + 1)^3 - x(x + 1)^2 = (x + 1)^2?$$

Solution: This is very much a trick question because *ALL* real numbers satisfy this equation:

$$(x + 1)^3 - x(x + 1)^2 = (x + 1)^2(x + 1 - x) = (x + 1)^2 \cdot 1 = (x + 1)^2.$$

Such equations are called *identities*. Students who ending up with $0 = 0$ and concluded all $x \in \mathbb{R}$ are solutions did not get full marks. This is a logical error. For example, that logic allows us to conclude all numbers are equal their negative:

$$\begin{aligned} x &= -x \\ \implies 0 &= 0 \\ \times_0 & \\ \implies x &= -x \text{ for all } x \in \mathbb{R}. \end{aligned}$$

4. Show that if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are distinct points on the curve

$$y = mx + c,$$

that the slope of PQ is equal to m .

Solution: Points on the curve satisfy the equation of the curve so we have

$$y_1 = mx_1 + c \text{ and } y_2 = mx_2 + c,$$

so we are finding the slope between $(x_1, mx_1 + c)$ and $(x_2, mx_2 + c)$, which is:

$$m_{PQ} = \frac{mx_2 + c - (mx_1 + c)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m,$$

where the points being distinct² means we could divide above and below by $x_2 - x_1$.

That is the slope between any two points on this curve have the same as any other pair — that is $y = mx + c$ is the equation of a line.

Students who assumed that $y = mx + c$ is a line... I guess this is a reasonable assumption (based on ordinary level leaving cert maths) but the point was rather to prove this.

5. Suppose that a function f is given by an infinite sum:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

Assuming that term-by-term differentiation is possible for this infinite sum, given that $a_0 = 1$, find the general term a_n of a non-zero sequence of real numbers $(a_0, a_1, a_2, a_3, \dots)$ such that the derivative of f is equal to f itself:

$$f'(x) = f(x).$$

Solution: This also assumes that two ‘infinite polynomials’ aka power series are equal if and only if all their coefficients are equal. Let us find an expression for $f'(x)$:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$

²in general $P(x_1, y_1) \neq Q(x_2, y_2)$ does not imply $x_1 \neq x_2$, but here it does because if $x_1 = x_2$ then $y_1 = y_2$ and $P = Q$

We see that to equal $f(x)$, the coefficients of x^{n-1} in $f(x)$ and $f'(x)$ must be equal:

$$a_{n-1} = n \cdot a_n \implies a_n = \frac{a_{n-1}}{n}.$$

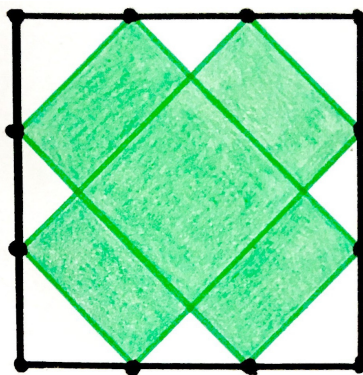
At this points students probably had to use this *recurrence* to find the first few values and hopefully spot a pattern:

$$a_0 = 1, \quad a_1 = \frac{1}{1} = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1/2}{3} = \frac{1}{1 \cdot 2 \cdot 3}, \quad a_4 = \frac{1/(1 \cdot 2 \cdot 3)}{4} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}.$$

Students with this much could reasonably write down:

$$a_n = \frac{1}{1 \cdot 2 \cdot 3 \cdots n} = \frac{1}{n!}.$$

6. The dots are equally spaced. What fraction of the square is shaded?



Solution: This question was statistically the easiest.

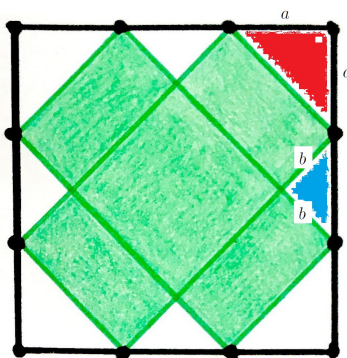


Figure 1: Let a be the common equal distance, and b the side-lengths in the blue isocles triangle.

The area of the red triangle is simply:

$$A_r = \frac{1}{2} \text{base} \cdot h = \frac{1}{2} a^2.$$

Using Pythagoras in the small blue triangle; and it's area:

$$a^2 = 2b^2 \implies b^2 = \frac{1}{2}a^2 \implies A_b = \frac{1}{2}b^2 = \frac{1}{4}a^2.$$

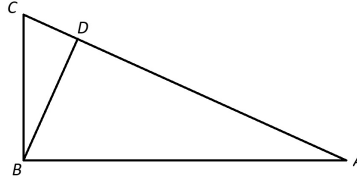
Therefore the area of the unshaded region is:

$$4 \cdot A_r + 4 \cdot A_b = 2a^2 + a^2 = 3a^2.$$

The total area is $(3a)^2 = 9a^2$, therefore the shaded area is $9a^2 - 3a^2 = 6a^2$ and the fraction shaded is:

$$\frac{6a^2}{9a^2} = \frac{2}{3}.$$

7. Consider a triangle $\triangle ABC$ with $\angle ABC$ a right-angle, and D a point on $[AC]$ such that $[BD]$ is perpendicular to $[AC]$:



Prove that

$$\sin^2(\angle BAC) = \frac{|CD|}{|AC|}.$$

Solution: It is possible to produce extremely convoluted solutions to this... on the other hand:

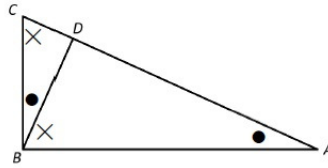


Figure 2: Note $\angle ABD = 90^\circ - \angle BAC$ and so $\angle CBD = \angle BAC$.

From this observation:

$$\begin{aligned} \sin^2(\angle BAC) &= \sin(\angle BAC) \cdot \sin(\angle BAC) \\ &= \sin(\angle BAC) \cdot \sin(\angle CBD) \\ &= \frac{|BC|}{|AC|} \cdot \frac{|CD|}{|BC|} = \frac{|CD|}{|AC|}. \end{aligned}$$

8. Consider three decks of playing cards, one belonging to Alice, one belonging to Bob, and one belonging to a referee. Alice chooses, not at random, their 21 favourite cards from their deck. Bob chooses, again not at random, their favourite 36 cards from their deck. The referee takes one card at random from their own deck. Let A be the event that the referee chose one of Alice's favourite cards, and B the event that the referee chose one of Bob's favourite cards.

- (a) Find $\mathbb{P}[A]$, the probability that the referee chose one of Alice's favourite cards.
- (b) Find $\mathbb{P}[B]$.
- (c) Recalling that Alice and Bob did *not* choose their favourite cards *at random*, find the maximum possible value of $\mathbb{P}[A \cap B]$, the probability that the referee chose a card that was a favourite of BOTH Alice AND Bob.
- (d) Similarly, find the minimum possible value of $\mathbb{P}[A \cap B]$.

Solution:

- (a) & (b) This is the straightforward

$$\mathbb{P}[A] = 21/52; \quad \mathbb{P}[B] = \frac{36}{52}.$$

- (c) Let us note that:

$$\mathbb{P}[A \cap B] = \frac{\# \text{ cards that are in both Alice AND Bob's favourites}}{52}.$$

So we want to make the number of cards to be in both Alice AND Bob's favourites to be as large as possible. This is achieved by supposing that ALL of Alice's favourites are also favourites of Bob's:

$$\max \quad \mathbb{P}[A \cap B] = \frac{21}{52}.$$

- (d) In this case we want as few cards in common between Alice and Bob. They have $21+36=57$ cards chosen in total so at least five have to be in both of their favourites:

$$\min \quad \mathbb{P}[A \cap B] = \frac{5}{52}.$$

Alternatively use "inclusion-exclusion":

$$|A \cup B| = |A| + |B| - |A \cap B| \implies |A \cap B| = |A| + |B| - |A \cup B|,$$

and to make this as small as possible you take away, $|A \cup B|$, as much as possible. We can achieve $|A \cup B| = 52$ — all cards are either a favourite of Alice or of Bob — and this leaves:

$$\min \quad |A \cap B| = 21 + 36 - 52 = 5.$$

9. In rugby, teams T_1 and T_2 can score three points with a penalty or drop goal, five points with an unconverted try, and seven points with a converted try. If the order of scores is not important, how many different ways could a match end with both teams scoring 20 points?

Note, T_1 with a drop goal and T_2 with a penalty is a ‘different’ 3-3 to the 3-3 where both T_1 and T_2 score a penalty.

Solution: Let us focus on one team at a time. Let C be the number of converted tries, T be the number of unconverted tries, P be the number of penalties, and D the number of drop goals. The total number of points we want for each team is:

$$7C + 5T + 3P + 3D \stackrel{!}{=} 20.$$

We can break into cases:

- (a) Suppose $C = 0$ so we consider:

$$5T + 3P + 3D = 20.$$

- i. If $T = 0$, $3P + 3D = 20$ has no solutions.
- ii. If $T = 1$, $3P + 3D = 15$. We get *six* solutions, one for each $P = 0, 1, \dots, 5$.
- iii. If $T = 2$ or $T = 3$, $3P + 3D = 10$ or $3P + 3D = 5$ have no solutions.
- iv. If $T = 4$ we have *one* solution.

- (b) Suppose $C = 1$ so we consider:

$$5T + 3P + 3D = 13.$$

- i. If $T = 0$ or $T = 1$ we have no solutions.
- ii. If $T = 2$ we have $3P + 3D = 3$ which has *two* solutions, one for $P = 0$, another for $P = 1$.

- (c) Suppose $C = 2$ so we consider:

$$5T + 3P + 3D = 6.$$

We must have $T = 0$ and then we get *three* solutions, one for each $P = 0, 1, 2$.

Altogether then there are $6 + 1 + 2 + 3 = 12$ solutions for T_1 . Therefore for T_1 vs T_2 there are $12 \times 12 = 144$ possible ways to make up a 20-20 draw.

10. There are three money boxes, labelled “Euro”, “Sterling”, and “Mixed”. You cannot look inside a box, but can press a button on it to remove a coin from that box.

One of the boxes has euro coins in it, another has sterling coins, and in the third a mixture of the two. But ALL three money boxes are labeled incorrectly. What is the minimum number of coins you can remove to work out which box is which?

Solution: The key is to understand that ALL the boxes are wrong.

In particular, the box labelled mixed is either sterling or euro. Get one coin out.

- (a) If it is euro, that box is Euro. The box labelled Sterling cannot be Sterling, so it must be Mixed. The Mixed box is Sterling.
- (b) If the one coin out is Sterling, that box is Sterling. The box labelled Euro cannot be Euro, so it must be Mixed. The Mixed box is Euro.

This means you can figure out all the boxes by removing ONE coin.